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Supergravity backgrounds for deformations of $\text{AdS}_n \times S^n$ supercoset string models

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Abstract

We consider type IIB supergravity backgrounds corresponding to the deformed $\text{AdS}_n \times S^n \times T^{10-2n}$ supercoset string models of the type constructed in arXiv:1309.5850 [2] which depend on one deformation parameter κ . In $\text{AdS}_2 \times S^2$ case we find that the deformed metric can be extended to a full supergravity solution with non-trivial dilaton, RR scalar and RR 5-form strength. The solution depends on a free parameter a that should be chosen as a particular function of κ to correspond to the deformed supercoset model. In $\text{AdS}_3 \times S^3$ case the full solution supported by the dilaton, RR scalar and RR 3-form strength exists only in the two special cases, $a = 0$ and $a = 1$. We conjecture that there may be a more general one-parameter solution supported by several RR fields that for particular $a = a(\kappa)$ corresponds to the supercoset model. In the most complicated deformed $\text{AdS}_5 \times S^5$ case we were able to find only the expressions for the dilaton and the RR scalar. The full solution is likely to be supported by a combination of the 5-form and 3-form field strengths. We comment on the singularity structure of the resulting metric and exact dilaton field.

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1. Introduction

Integrability of string sigma model is a key feature that allows to determine the string spectrum in non-trivial curved backgrounds. The study of integrable deformations of the most-symmetric $\text{AdS}_5 \times \text{S}^5$ model underlying AdS/CFT correspondence [1] is thus an important avenue of research that may also shed light on hidden symmetries of dual gauge theories. Recently, a novel one-parameter integrable deformation of the $\text{AdS}_5 \times \text{S}^5$ supercoset model was constructed in [2] (see also [3–8]). This model generalizes some previously known low-dimensional bosonic integrable models [9–13].

The corresponding target space type IIB supergravity background has no space-time supersymmetry and the bosonic isometry is reduced from $SO(2, 4) \times SO(6)$ to its Cartan subgroup $[SO(2)]^3 \times [SO(2)]^3$, i.e. most of the symmetry of the original $\text{AdS}_5 \times \text{S}^5$ space becomes hidden (or “ q -deformed”). Starting with a specific parametrization of the bosonic part of the deformed supercoset model [2] the corresponding 10d metric and B -field were found explicitly in [3]. However, extracting the associated RR field strengths that should promote the deformed metric to an exact supergravity solution from the fermionic part of the supercoset action turns out to be challenging even in the simpler low-dimensional $\text{AdS}_2 \times \text{S}^2$ and $\text{AdS}_3 \times \text{S}^3$ models [4].

Our aim here will be to find the deformed $\text{AdS}_n \times \text{S}^n$ type IIB backgrounds by (i) starting with the deformed metric as given by the bosonic part of the supercoset model and (ii) solving the supergravity equations directly to find the expressions of all other fields required to support this metric as an exact solution. Finding “matter” fields supporting a given metric via Einstein equations is not a standard GR problem; the solution may not exist or, if it exists, it may not be unique. The present case is complicated also by the absence of supersymmetry and non-abelian isometries. We shall see that the solutions will have a rather unusual feature: while the string-frame metric is a direct sum of the deformed AdS_n and S^n parts, this will no longer be so for the dilaton and the RR fields – they will not factorize and thus “tie” the AdS_n and S^n parts together (as what fermion part of supercoset model does).

Having found a supergravity solution with the required deformed $\text{AdS}_n \times \text{S}^n$ metric, one is still to decide if it is the one that actually corresponds to the integrable deformed supercoset model of [2]. As we shall see below, the solution for the dilaton and RR fluxes supporting a given deformed metric is not unique: in $\text{AdS}_2 \times \text{S}^2$ case there is a one-parameter a -family of solutions, and the same is expected to be the case also in the $\text{AdS}_3 \times \text{S}^3$ and $\text{AdS}_5 \times \text{S}^5$ cases. One is then to choose a as a function of the deformation parameter κ ² in order to match the supercoset model. This choice may be aided by consideration of the two special limits discussed in [4]:

- (i) $\kappa = \infty$ or “maximal deformation limit”: in this case the deformed $\text{AdS}_n \times \text{S}^n$ supercoset model becomes T-dual to “double Wick rotation” of the undeformed $\text{AdS}_n \times \text{S}^n$ model, i.e. it has $d\text{S}_n \times \text{H}^n$ target space supported by an imaginary n -form RR flux;
- (ii) $\kappa = i$ (combined with a rescaling of coordinates and string tension) or “pp-wave limit”: in this case the target-space metric becomes of pp-wave type and the problem of finding the supporting dilaton and fluxes simplifies.

² We shall follow [3] and use $\kappa = \frac{2\eta}{1-\eta^2}$ as the deformation parameter, where η is the parameter used in [2].

We shall start in Section 2 by finding a one-parameter type IIB solution with the metric being that of the κ -deformation of the $\text{AdS}_2 \times \text{S}^2 \times \text{T}^6$ one [4]. It corresponds to a solution of 4d supergravity obtained by compactification on 6-torus with only the dilaton, RR scalar and the RR 2-form being non-trivial. Guided by the two special limits mentioned above we shall argue that for a special value of the free parameter $a = a(\kappa) = \kappa^{-1}\eta = \kappa^{-2}(\sqrt{\kappa^2 + 1} - 1)$ the resulting background should correspond to the κ -deformation of the $\text{AdS}_2 \times \text{S}^2$ supercoset model.

In Section 3 we shall consider the κ -deformation of the $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ space supported by the RR 3-form flux. Compactifying on 4-torus we shall use the truncated 6d action containing the metric, dilaton, RR scalar and RR 3-form field. Starting with the κ -deformed $\text{AdS}_3 \times \text{S}^3$ metric [4] we will find again a one-parameter family of solutions of the three scalar equations. However, only two special members of this family (with $a = 0$ and $a = 1$) will have extensions to solutions of the full set of 6d supergravity equations if only one RR 3-form field is assumed to be non-zero. The existence of the complete solution with an arbitrary parameter a (that may be chosen again as $a(\kappa)$ to match the deformed supercoset model) appears to require more RR field strengths to be non-zero, a possibility which remains to be studied. We shall also present the analogs of the $a = 0$ and $a = 1$ solutions in the case of two-parameter (κ_+, κ_-) deformation of the $\text{AdS}_3 \times \text{S}^3$ supercoset [14] with the metric corresponding to the two-parameter Fateev model [10] for deformations of AdS_3 and S^3 .

Guided by the low-dimensional examples, in Section 5 we shall address the problem of promoting the κ -deformed $\text{AdS}_5 \times \text{S}^5$ metric and the B -field found in [3] to the full type IIB supergravity solution. An additional complication is that the 10d metric (and thus also other background fields) contains a non-trivial dependence on two extra angular coordinates. We will present two special solutions to the equations for the dilaton and the RR scalar which are the counterparts of the $a = 0$ and $a = 1$ solutions in the $\text{AdS}_3 \times \text{S}^3$ case. Here we will not be able to find the corresponding 5-form flux and it appears likely that the full solution should exist only when also the RR 3-form flux is non-zero.

Some comments on the singularity properties of the deformed $\text{AdS}_n \times \text{S}^n$ backgrounds will be included in Section 5. In Appendix A we will give the form of the relevant supergravity equations in different dimensions and discuss truncations of the 10d supergravity action. In Appendix B we will review the algebraic Rainich conditions on Maxwell stress tensor in 4 dimensions.

2. Deformation of $\text{AdS}_2 \times \text{S}^2$

In this section we shall extend the metric of the deformation of the $\text{AdS}_2 \times \text{S}^2 \times \text{T}^6$ space [4] to a 10d type IIB supergravity solution. Similarly to the undeformed background [15], this solution is a direct 10d lift of the corresponding solution of 4d supergravity obtained by compactification on 6-torus: the 5-form field strength F_5 is given by the product of the non-trivial 2-form field strength F_2 in 4 dimensions and the canonical hermitian 3-form of T^6 .³ The background fields (metric, dilaton, RR scalar and 1-form potential) will depend on a free parameter a . We shall conjecture that for a special choice of $a = a(\kappa)$ the resulting background should correspond to the superstring sigma model which is the κ -deformation of the $\text{AdS}_2 \times \text{S}^2$ supercoset model based on $\text{PSU}(1, 1|2)/\text{U}(1) \times \text{U}(1)$. As a check, we shall show that in the special limits of $\kappa = \infty$

³ As described in Appendix A, this background can be embedded into type IIA supergravity (see (A.15), (A.18)) or in type IIB SUGRA (see (A.19)). These two embeddings are related by T dualities, and other duality frames might be interesting as well.

($a = 0$) and $\kappa = i$ ($a = 1$) we indeed reproduce the expressions expected from the deformed supercoset construction.

2.1. One-parameter family of solutions

Let us recall that the $\text{AdS}_2 \times \text{S}^2 \times \text{T}^6$ solution can be obtained as a limit of 10d type IIB solution describing four intersecting stacks of D3-branes (see, e.g., [15] and refs. there). Upon reduction on T^6 to four dimensions it is supported by a 2-form field strength F_2 . The compactification of type II supergravity to four dimensions on T^6 in general contains a large number of scalar and vector fields, some of which describe the deformations of the compact space. Since by construction the deformation acts only on the supercoset part of the geometry we may assume that the fields that should be non-vanishing are not related to T^6 . The minimal choice is the metric, dilaton, the RR scalar C and the vector A (with $F_2 = dA$ as its field strength; the latter may represent several identified components of the 10d fields, cf. Appendix A). The Lagrangian for 4d supergravity restricted to these fields is⁴

$$\mathcal{L}_4 = e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 \right] - \frac{1}{4} F_{mn} F^{mn} - \frac{1}{2} (\partial C)^2. \quad (2.1)$$

The simplest solution is the $\text{AdS}_2 \times \text{S}^2$ Bertotti–Robinson one with Φ and C being constant and

$$ds^2 = L^2 \left[-(1 + \rho^2) dt^2 + \frac{d\rho^2}{1 + \rho^2} \right] + L^2 \left[(1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right], \quad (2.2)$$

$$F_2 = 2L(c_1 d\rho \wedge dt + c_2 dr \wedge d\varphi), \quad c_1^2 + c_2^2 = 1. \quad (2.3)$$

Here c_1, c_2 are reflecting the freedom of $\text{U}(1)$ electromagnetic duality rotations.

Our aim will be to find Φ, C and F_2 that promote the deformed $\text{AdS}_2 \times \text{S}^2$ metric [4]

$$ds^2 = \frac{L^2}{1 - \kappa^2 \rho^2} \left[-(1 + \rho^2) dt^2 + \frac{d\rho^2}{1 + \rho^2} \right] + \frac{L^2}{1 + \kappa^2 r^2} \left[(1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right] \quad (2.4)$$

to an exact solution of the theory (2.1). Here L is the (inverse) curvature scale (that we shall often set to 1 in what follows) and κ is the parameter of deformation away from the symmetric $\text{AdS}_2 \times \text{S}^2$ point. The Ricci tensor and the curvature scalar of the metric $g^A \oplus g^S$ in (2.4) can be written as

$$R_{ab}^A = -(1 + \kappa^2) \frac{1 + \kappa^2 \rho^2}{1 - \kappa^2 \rho^2} g_{ab}^A, \quad R_{ab}^S = (1 + \kappa^2) \frac{1 - \kappa^2 r^2}{1 + \kappa^2 r^2} g_{ab}^S \quad (2.5)$$

$$R = 4(1 + \kappa^2) \left(-\frac{1}{1 - \kappa^2 \rho^2} + \frac{1}{1 + \kappa^2 r^2} \right). \quad (2.6)$$

The equations of motion following from (2.1) are given in Appendix A; we shall focus first on the trace of the Einstein equation, the equation for the RR scalar and the equation for the dilaton that can be organized as (cf. (A.7))

$$R + 2\nabla^2\Phi + \frac{1}{2} e^{2\Phi} \partial_m C \partial^m C = 0, \quad \nabla^2 C = 0, \quad \nabla^2 (C^2 + 4e^{-2\Phi}) = 0. \quad (2.7)$$

⁴ In general, the action may contain also a term $\alpha C F_{mn} \tilde{F}^{mn}$ with some special constant α . It is possible to show that in the present case one should choose the identification of the fields such that $\alpha = 0$ as otherwise one will not get the undeformed $\text{AdS}_2 \times \text{S}^2$ background as a solution.

A way to solve this system is to consider first a particular limit: a small κ expansion combined with a particular rescaling of the coordinates

$$\kappa \rightarrow 0, \quad \text{with fixed } \kappa\rho, \kappa^{-1}t, \quad (2.8)$$

in which the S^2 part of the metric becomes undeformed while the deformation of the AdS_2 part remains non-trivial, i.e.

$$ds^2 = \frac{1}{1 - (\kappa\rho)^2} \left[-\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} \right] + \left[(1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right]. \quad (2.9)$$

The perturbative expansion in κ respecting the symmetry $\kappa \rightarrow \lambda\kappa$, $\rho \rightarrow \frac{\rho}{\lambda}$, $t \rightarrow \lambda t$ of the metric (2.9) should then be an expansion in powers of $\kappa\rho$:

$$e^{-2\Phi} = 1 + \sum (\kappa\rho)^n f_n(r), \quad C = \sum (\kappa\rho)^n g_n(r). \quad (2.10)$$

Substituting this into (2.7) and summing up the perturbative series, we find the most general solution corresponding to the metric (2.9). The solution depends on one free parameter a :

$$e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)}{1 - a^2(\kappa\rho)^2 + (\kappa\rho r)^2 - 2\kappa\sqrt{1 - a^2}\rho r}, \quad C = 2\sqrt{\frac{1}{a^2} - e^{-2\Phi}}. \quad (2.11)$$

Note that here the combination $C^2 + 4e^{-2\Phi}$ that should be a harmonic function according to (2.7) is simply a constant

$$C^2 + 4e^{-2\Phi} = \frac{4}{a^2}. \quad (2.12)$$

Going back to the general case of the metric (2.4) and requiring that the solution of (2.7) should have the same property (2.12) leads to a similar one-parameter solution for the scalar fields. It is then easy to find also the solution for the vector potential A^5

$$ds^2 = \frac{1}{1 - \kappa^2 \rho^2} \left[-(1 + \rho^2) dt^2 + \frac{d\rho^2}{1 + \rho^2} \right] + \frac{1}{1 + \kappa^2 r^2} \left[(1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right], \quad (2.13)$$

$$e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)}{P(\rho, r)}, \quad P(\rho, r) \equiv 1 + \kappa^2 [a^2(r^2 - \rho^2) - 2br\rho + r^2 \rho^2], \quad (2.14)$$

$$C = 2\sqrt{\frac{1}{a^2} - e^{-2\Phi}} = \frac{2}{a\sqrt{P(\rho, r)}} [\sqrt{1 - a^2} - \kappa\sqrt{1 + a^2\kappa^2}\rho r], \quad (2.15)$$

$$A = \frac{2}{\sqrt{P(\rho, r)}} [\sqrt{1 + a^2\kappa^2}(c_1\rho dt + c_2r d\varphi) + \kappa\sqrt{1 - a^2}(c_1r dt - c_2\rho d\varphi)],$$

$$b \equiv \frac{1}{\kappa} \sqrt{(1 - a^2)(1 + a^2\kappa^2)}, \quad c_1^2 + c_2^2 = 1. \quad (2.16)$$

⁵ One may check that the candidate F_2 implied by the form of the Maxwell stress tensor appearing on the right-hand side of Einstein's equations for the given metric and the scalar fields obeys the algebraic Rainich condition [16–19] (see Appendix B), i.e. there should indeed exist a vector field sourcing this geometry.

This solution depends on the parameter a and also on a trivial parameter c_1 reflecting again the freedom of U(1) electromagnetic duality which is the symmetry of the equations following from (2.1) (we may always assume that $c_1 = c_2 = \frac{1}{\sqrt{2}}$ without loss of generality).

Let us note that the solution for the scalar C is of course defined up to a constant. Using this the special solution corresponding to $a = 0$ can be written as⁶

$$a = 0: \quad e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)}{(1 - \kappa r \rho)^2}, \quad C = 0, \\ A = \frac{2}{1 - \kappa r \rho} [c_1(\rho + \kappa r)dt + c_2(r - \kappa \rho)d\varphi]. \quad (2.17)$$

Another special case corresponds to $a = 1$:

$$a = 1: \quad e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)}{1 + \kappa^2(r^2 - \rho^2 + r^2 \rho^2)}, \\ C = 2\sqrt{1 - e^{-2\Phi}} = -\frac{2\kappa\sqrt{1 + \kappa^2}}{\sqrt{1 + \kappa^2(r^2 - \rho^2 + r^2 \rho^2)}} \rho r, \\ A = \frac{2\sqrt{1 + \kappa^2}}{\sqrt{1 + \kappa^2(r^2 - \rho^2 + r^2 \rho^2)}} (c_1 \rho dt + c_2 r d\varphi). \quad (2.18)$$

2.2. Symmetries and limits of the solution

The free parameter a should be fixed in order to establish a relation to the supercoset model which depends just on κ . To understand possible dependence of a on κ let us now discuss some properties and limits of the solution in (2.13)–(2.16). It turns out that it is invariant under certain sequences of dualities and analytic continuations:

A. T-dualities

1. Perform T-dualities along t and φ directions.
2. Analytically continue the new coordinates $(t, \varphi) \rightarrow i(t, \varphi)$ and rescale $(\rho, r) \rightarrow \ell(\rho, r)$, $\ell \equiv \kappa^{-1}$.
3. Replace the 2-form potential, appearing after the T-dualities, by an axion via tensor-scalar duality in 4d.
4. Rescale the dilaton, the axion, and the Maxwell field to make $e^{-2\Phi} = 1$ when $r = \rho = 0$. Then the resulting geometry coincides with (2.13)–(2.16) upon the identification

$$a \rightarrow i\kappa a, \quad \ell \rightarrow \kappa. \quad (2.19)$$

B. Inversion of coordinates

1. Rewrite (2.13)–(2.16) in terms of $x \equiv 1/\rho$ and $y \equiv 1/r$.
2. Define $\ell \equiv \kappa^{-1}$ and $\tilde{L} = -i\ell L$ (we restore the overall scale L in the metric (2.4)).
3. Rescale the dilaton, the axion, and the Maxwell field to make $e^{-2\Phi} = 1$ when $r = \rho = 0$. The resulting geometry coincides with (2.13)–(2.16) upon the identification (2.19).

⁶ Note that the infinite shift of the RR scalar effectively makes $C^2 + 4e^{-2\Phi}$ a function of the coordinates. Even for $a \neq 0$ one can perform a constant shift to have $C = 0$ for $\kappa = 0$, making $C^2 + 4e^{-2\Phi}$ somewhat complicated. For $a = 0$ the shift is required.

The transformation B has an implication for the large κ limit of (2.13)–(2.16): if we want to send κ to infinity while keeping the metric finite, then L/κ must remain fixed. In the transformation B this corresponds to sending $\ell = \kappa^{-1}$ to zero while keeping \tilde{L} fixed; Eq. (2.19) then implies that such limit leads to imaginary fluxes unless $a = 0$. To see this more explicitly, let us consider the large κ limit of (2.13)–(2.16) for $a \neq 0$:

$$\begin{aligned} ds^2 &= \frac{L^2}{-\kappa^2 \rho^2} \left[-(1 + \rho^2) dt^2 + \frac{d\rho^2}{1 + \rho^2} \right] + \frac{L^2}{\kappa^2 r^2} \left[(1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right] \\ e^{-2\Phi} &= -\frac{\gamma^2 \kappa^2 \rho^2 r^2}{a^2(r^2 - \rho^2) - 2br\rho + r^2 \rho^2} \equiv -\frac{\gamma^2 \kappa^2 \rho^2 r^2}{P(\rho, r)}, \quad b = a\sqrt{1 - a^2} \\ C &= 2\sqrt{\frac{\gamma^2}{a^2} - e^{-2\Phi}} = \frac{\gamma}{\sqrt{P(\rho, r)}} \kappa \rho r, \\ A &= \frac{2L\gamma}{\sqrt{P(\rho, r)}} [a(c_1 \rho dt + c_2 r d\varphi) + \sqrt{1 - a^2}(c_1 r dt - c_2 \rho d\varphi)]. \end{aligned} \quad (2.20)$$

Here we kept all coordinates fixed and rescaled the exponent of the dilaton and the RR fluxes by a free parameter γ . It is clear that no real value of this parameter makes e^Φ positive while keeping C real. This argument breaks down only for $a = 0$, when the expression for C is to be modified by an infinite constant shift. For $a = 0$ we get

$$\begin{aligned} ds^2 &= \frac{L^2}{-\kappa^2 \rho^2} \left[-(1 + \rho^2) dt^2 + \frac{d\rho^2}{1 + \rho^2} \right] + \frac{L^2}{\kappa^2 r^2} \left[(1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right] \\ e^{-2\Phi} &= -\gamma^2 \kappa^2, \quad C = 0, \quad A = \frac{2L\gamma}{(r\rho)^2} (c_1 r dt - c_2 \rho d\varphi). \end{aligned} \quad (2.21)$$

Setting $L = i\kappa$, $\gamma = 1/L$, we find $\text{AdS}_2 \times \text{S}^2$ in the inverted coordinates. This suggests that the parameter a should vanish in this large κ limit.

A different way of taking the large κ limit of (2.13)–(2.16) is found by rescaling the coordinates and L as follows (the variables with tildes are to be kept fixed)

$$t = \frac{\tilde{t}}{L}, \quad \varphi = \frac{\tilde{\varphi}}{L}, \quad \rho = \frac{\tilde{\rho}}{L}, \quad r = \frac{\tilde{r}}{L}, \quad \kappa = L\tilde{\kappa}, \quad \kappa \rightarrow \infty, \quad L \rightarrow \infty. \quad (2.22)$$

Taking the limit $\kappa, L \rightarrow \infty$ in (2.13) we then get (omitting tildes) [6]

$$ds^2 = \frac{1}{1 - \kappa^2 \rho^2} (-dt^2 + d\rho^2) + \frac{1}{1 + \kappa^2 r^2} (d\varphi^2 + dr^2) \quad (2.23)$$

The T-dualities in t and φ applied to this metric give $\text{dS}_2 \times \text{H}_2$ space which is naturally a solution with a constant dilaton. Then the simplest choice for the dilaton that represents a solution together with the metric (2.23) should be [6]

$$e^{-2\Phi} = (1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2). \quad (2.24)$$

On the other hand, in the limit (2.22) the dilaton in (2.14) becomes

$$e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)}{1 + (\kappa a)^2(r^2 - \rho^2) - 2a\sqrt{1 - a^2}\kappa^2 r\rho}. \quad (2.25)$$

To match (2.24) we should thus set $a = 0$. This suggests that if a is a function of κ then one should have $a(\kappa \rightarrow \infty) \rightarrow 0$.

Another useful limit corresponds to setting $\kappa = i$ in (2.13)–(2.16) (see [4]). Then metric (2.4), (2.13) becomes flat, and the dilaton (2.14) takes the form

$$e^{-2\Phi} = \frac{(1 + \rho^2)(1 - r^2)}{(1 + \rho^2)(1 - r^2) - (1 - a^2)(\rho + ir)^2}. \quad (2.26)$$

This expression is real only if $a = 1$ suggesting that one should have $a(\kappa \rightarrow i) \rightarrow 1$. In this case the dilaton becomes constant as appropriate for a “minimal” choice of the dilaton solution in the case a flat metric. There is also a special way of taking this $\kappa = i$ limit by combining it with a rescaling of the coordinates

$$t = \frac{x^+}{\varepsilon} - \varepsilon x^-, \quad \varphi = \frac{x^+}{\varepsilon} + \varepsilon x^-, \quad \varepsilon^2 = \kappa^2 + 1 \rightarrow 0. \quad (2.27)$$

This leads to a pp-wave 4d metric [4]. In this limit the F_2 flux in (2.16) diverges unless again $a = 1$.

2.3. Choice of $a(\kappa)$

In the previous subsection we discussed the natural values of a for the two special values of κ :

- (i) $\kappa = \infty$: in this limit the metric is related (T-dual) to an analytic continuation of $\text{AdS}_2 \times \text{S}^2$ and the simplest choice is to set $a(\infty) = 0$.
- (ii) $\kappa = i$: the assumption that Φ and C should remain real within the family of solutions parametrized by a implies that $a(\kappa)$ should satisfy $a(i) = 1$.

Let us now propose a particular function $a(\kappa)$ which has the required limits and is also consistent with the structure of the supercoset action. The deformed supercoset action of [2] depends naturally on combination of the projectors $\kappa P_2 + \eta(P_1 - P_3)$ where

$$\eta = \frac{\sqrt{\kappa^2 + 1} - 1}{\kappa}, \quad (2.28)$$

and P_k are projectors on the supergroup elements with i^k charge under Z_4 transformations. The string sigma model action and thus the background fields should then contain the two parameters κ and η entering simply as a ratio. We conjecture that the solution (2.13)–(2.16) with $a(\kappa)$ given by

$$a(\kappa) = \frac{\eta}{\kappa} = \frac{\sqrt{\kappa^2 + 1} - 1}{\kappa^2} = \frac{1}{\sqrt{\kappa^2 + 1} + 1} \quad (2.29)$$

should correspond to the $\text{AdS}_2 \times \text{S}^2$ supercoset model. Then

$$a(0) = \frac{1}{2}, \quad a(i) = 1, \quad a(\infty) = 0, \quad (2.30)$$

in agreement with the above discussion of the two special limits.

3. Deformation of $\text{AdS}_3 \times \text{S}^3$

In the previous section we constructed a supergravity solution that should represent the background underlying the κ -deformed $\text{AdS}_2 \times \text{S}^2$ supercoset model. The important ingredient was

the existence of a one-parameter family of solutions with the free parameter a which was then fixed to be a specific function of κ to match the corresponding limits of the supercoset construction.

In this section we shall attempt to follow the same strategy for the κ -deformation of $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ space supported by RR 3-form flux. Compactifying on 4-torus we shall use the effective 6d action containing the dilaton Φ , RR scalar C and RR 3-form field strength F_3 . Starting with the κ -deformed $\text{AdS}_3 \times \text{S}^3$ metric [4] we will find again a one-parameter family of solutions of the three scalar equations of the 6d theory. It will turn out, however, that only two members of this family – the analogs of the $a = 0$ and $a = 1$ solutions in (2.17) and (2.18) – can be extended to solutions of the full set of 6d equations if one assumes that in addition to Φ and C only one RR 3-form is non-zero.

It is likely that there should exist a more general solution (with an additional F_5 field in 10d or an extra F_3 field in 6d) parametrized by an arbitrary a that should match the supercoset model for a special choice of $a = a(\kappa)$.

3.1. One-parameter family of solutions of the scalar equations

We shall start with the following “minimal” 6d Lagrangian representing a reduction and truncation of type IIB 10d supergravity on 4-torus. As discussed in Appendix A, consistent truncation leads to the Lagrangian

$$\mathcal{L}_6 = e^{-2\Phi} \left[R + 4(\nabla\Phi)^2 \right] - \frac{1}{12} F_{mnp} F^{mnp} - \frac{1}{2} (\partial C)^2, \quad (3.1)$$

supplemented by an additional constraint (A.13) (see Appendix A):

$$\frac{1}{12} F_{mnp} F^{mnp} + \frac{1}{2} (\partial C)^2 = 0. \quad (3.2)$$

The equations of motion for (3.1) are also given in Appendix A. The simplest solution is $\text{AdS}_3 \times \text{S}^3$ supported by the self-dual F_{mnk} (with Φ and C being trivial). We will be interested in finding a solution for which the metric is given by the κ -deformed $\text{AdS}_3 \times \text{S}^3$ metric implied by the supercoset construction [3,4]

$$ds^2 = \frac{1}{1 - \kappa^2 \rho^2} \left[-(1 + \rho^2) dt^2 + \frac{d\rho^2}{1 + \rho^2} \right] + \rho^2 d\chi^2 + \frac{1}{1 + \kappa^2 r^2} \left[(1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right] + r^2 d\psi^2. \quad (3.3)$$

As in the previous section we shall first focus on the three scalar equations: the trace of Einstein’s equation, the RR scalar one and the dilaton one that can be organized as (cf. (A.7) and (2.7))

$$R + 2\nabla^2\Phi + \frac{1}{2} e^{2\Phi} \partial_m C \partial^m C = 0, \quad \nabla^2 C = 0, \quad \nabla^2 \left(\frac{1}{2} C^2 + e^{-2\Phi} \right) = 0. \quad (3.4)$$

We begin by first solving them perturbatively in the small κ limit with $\kappa\rho, \kappa^{-1}t$ being fixed as in (2.8), (2.9) when the metric becomes

$$ds^2 = \frac{1}{1 - \kappa^2 \rho^2} \left[-\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} \right] + \rho^2 d\chi^2 + (1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} + r^2 d\psi^2. \quad (3.5)$$

Expanding in powers of $\kappa\rho$ as in (2.10) we find a unique solution (regular at $\kappa\rho = 0$) which depends on one-parameter a :

$$e^{-2\Phi} = \frac{1 - \kappa^2 \rho^2}{P_2(\rho, r)},$$

$$P_2(\rho, r) \equiv [1 - \kappa^2(\rho r)^2]^2 + 2a^2(2r^2 - 1)(\kappa\rho)^2 - a^2(2r^2 - a^2)(\kappa\rho)^4, \quad (3.6)$$

$$C = \frac{\sqrt{2}}{a\sqrt{(1 - a^2)P_2(\rho, r)}} [1 - 2a^2 + (\kappa\rho)^2(r^2 - a^2)]. \quad (3.7)$$

This small κ solution can be generalized to the arbitrary κ solution of the three scalar equations (3.4):

$$e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)}{P_2(\rho, r)}, \quad (3.8)$$

$$P_2(\rho, r) \equiv [1 - \kappa^2(\rho r)^2]^2 + 2a^2\kappa^2[r^2 - \rho^2 + 2(\rho r)^2] + 2\kappa^4 a^2(\rho r)^2(r^2 - \rho^2 - 2) + \kappa^4 a^4(\rho^2 + r^2)^2,$$

$$C = \sqrt{\frac{1}{2a^2(1 - a^2)}} - 2e^{-2\Phi}$$

$$= \frac{\sqrt{2}}{a\sqrt{(1 - a^2)P_2(\rho, r)}} [1 - \kappa^2(\rho r)^2 - a^2(2 - \kappa^2\rho^2 + \kappa^2 r^2)]. \quad (3.9)$$

The same choice (2.29) for $a(\kappa)$ as in the $\text{AdS}_2 \times \text{S}^2$ case then gives us a solution which is consistent with both $\kappa = \infty$ and $\kappa = i$ limits.

It is interesting to note a relation between the quadratic polynomial P in the deformed $\text{AdS}_2 \times \text{S}^2$ solution (2.13)–(2.16) and the quartic polynomial P_2 in (3.9). If we define the analog of $P \equiv P_-$ in (2.14) with⁷ $b \rightarrow -b$ as P_+ then we observe that P_2 can be written as a product of P_+ and P_- , i.e.

$$P_2 = P_+ P_-, \quad P = P_-,$$

$$P_{\pm} \equiv 1 + \kappa^2 a^2(r^2 - \rho^2) \pm 2\kappa\sqrt{(1 - a^2)(1 + a^2\kappa^2)}r\rho + \kappa^2 r^2 \rho^2. \quad (3.10)$$

Attempting to extend this one-parameter solution of the scalar equations to a solution of the full set of 6d equations following from (3.1)–(3.2) using an ansatz-based approach suggests that this is possible only for the special values 0 and 1 of the parameter a . We shall also see another indication of this obstruction from the algebraic constraints on the 3-form stress tensor discussed in the next subsection.

3.2. Existence of a field strength for a given stress tensor: Rainich conditions

The question we are facing is how to find a 3-form flux supporting (together with the scalar fields) a given metric through the Einstein equation, i.e. how to find a solution for the flux given

⁷ Changing the sign of b maps (2.13)–(2.16) into another solution provided one also changes the relative sign of the two terms in the 1-form field in (2.16).

a specific form of its stress tensor. In general, the question is when some field configuration (i.e. some metric as well as other fields) can be sourced by an n -form field in $d = 2n$ dimensions.

In the four-dimensional Einstein–Maxwell theory this question was addressed long ago [16–19]: in order for some stress tensor T_{mn} implied by Einstein’s equations to be generated by a Maxwell field strength T_{mn} should be traceless and also its third power should be traceless as well (a brief derivation of this fact is given in [Appendix B](#)).

Here we find the analogous conditions in six dimensions (the generalization to higher dimensions is also straightforward). Let us consider the stress tensor of a 3-form field strength

$$T_m{}^n = F_{mkl} F^{kln} - \frac{1}{6} \delta_m^n F_{skl} F^{kls}. \quad (3.11)$$

Direct calculation shows that it satisfies

$$\text{tr } T = 0, \quad \text{tr } T^3 = 0, \quad \text{tr } T^5 = 0. \quad (3.12)$$

Thus given a six-dimensional background (metric, dilaton, etc.) and computing the effective stress tensor T_{mn} in the right-hand side of the Einstein equation that should be representing the contribution of the 3-form field, this T_{mn} should satisfy Eq. (3.12) in order for F_{mnk} to exist. This is a necessary condition, which in general may not be a sufficient one.

Some additional constraints may appear for special choices of the field strength. For example, for an (imaginary)-self-dual field strength we find that

$$4d: \quad T_m{}^n = 0, \quad 6d: \quad T^2 = \frac{1}{6} \text{tr } T^2. \quad (3.13)$$

A similar analysis implies that the necessary conditions that some 10d symmetric 2nd rank tensor may be the stress tensor of a 5-form field strength are

$$\text{tr } T = 0, \quad \text{tr } T^3 = 0, \quad \text{tr } T^5 = 0, \quad \text{tr } T^7 = 0, \quad \text{tr } T^9 = 0. \quad (3.14)$$

These relations hold, in particular, for $\text{AdS}_5 \times M^5$ solutions, where M^5 is an Einstein space.

3.3. Complete solutions

Starting with the metric (3.5) and the dilaton (3.6) and RR scalar (3.7) one can find explicitly the expected stress tensor for the 3-form RR field $F_3 = dC_2$

$$\begin{aligned} T_m{}^n &\equiv e^{-2\Phi} (R_m{}^n + 2\nabla_m \nabla^n \Phi) - \frac{1}{2} \left[\partial_m C \partial^n C - \frac{1}{2} \delta_m^n (\nabla C)^2 \right] \\ &= \frac{1}{4} \left(F_{mpq} F^{npq} - \frac{1}{6} \delta_m^n F_{spq} F^{spq} \right). \end{aligned} \quad (3.15)$$

Direct calculation shows that it satisfies the non-trivial (last two) relations in (3.12) only for the special values $a = 0, 1$ of the parameter in (3.7). Thus (3.7) can be supported by the 3-form flux only in these two special cases. It may be possible to go around this problem by allowing for two non-vanishing independent 3-form fields in the reduced 6d Lagrangian (3.1). We will not attempt to study this option here.

The corresponding small κ limit solutions (supplementing the metric (3.5)) are:

$$a = 0: \quad e^{-2\Phi} = \frac{1 - \kappa^2 \rho^2}{[1 - (\kappa \rho r)^2]^2}, \quad C = 0,$$

$$C_2 = \frac{\rho^2}{1 - (\kappa \rho r)^2} [dt + \kappa(1 - r^2)d\varphi] \wedge [d\chi + \kappa r^2 d\psi] - r^2 d\varphi \wedge d\psi, \quad (3.16)$$

$$a = 1: \quad e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)}{[1 - (\kappa \rho)^2(1 - r^2)]^2}, \quad C = 0,$$

$$C_2 = \frac{1}{1 - (\kappa \rho)^2(1 - r^2)} [\rho^2 dt \wedge d\chi - \kappa \rho^2(1 - r^2) dt \wedge d\varphi$$

$$+ \kappa(\rho r)^2 d\chi \wedge d\psi - r^2 d\varphi \wedge d\psi]. \quad (3.17)$$

These small- κ limit solutions can be extended to solutions with general κ by making ansätze that dress the solutions (3.16) and (3.17) with numerator and denominator functions of κ , r and ρ which become unity in the small κ limit. The resulting exact solutions are found to be (cf. (2.17), (2.18))

$$a = 0: \quad e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)}{[1 - (\kappa \rho r)^2]^2}, \quad C = 0,$$

$$C_2 = \frac{1}{1 - (\kappa \rho r)^2} [\rho^2(dt + \kappa d\varphi) \wedge (d\chi + \kappa r^2 d\psi)$$

$$- r^2(d\varphi - \kappa dt) \wedge (d\psi + \kappa \rho^2 d\chi)] \quad (3.18)$$

$$a = 1: \quad e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)}{[1 + \kappa^2(r^2 - \rho^2 + r^2 \rho^2)]^2}, \quad C = 0,$$

$$C_2 = \frac{\sqrt{1 + \kappa^2}}{1 + \kappa^2(r^2 - \rho^2 + r^2 \rho^2)} (\rho^2 dt \wedge d\chi + \kappa[r^2 - \rho^2 + (\rho r)^2] dt \wedge d\varphi$$

$$+ \kappa(\rho r)^2 d\chi \wedge d\psi - r^2 d\varphi \wedge d\psi). \quad (3.19)$$

There are also solutions with flipped signs of t , χ , φ , ψ .

Let us note that, as in the 4d case (2.3), the undeformed $\text{AdS}_3 \times \text{S}^3$ metric (i.e. (3.3) with $\kappa = 0$) can be supported by a one-parameter family of 2-form potentials

$$C_2 = \sqrt{2}(c_1 \rho^2 dt \wedge d\chi + c_2 r^2 d\varphi \wedge d\psi), \quad c_1^2 + c_2^2 = 1. \quad (3.20)$$

However, this freedom does not extend to the case of $\kappa \neq 0$ with non-trivial Φ and C . This is related to a different structure of the “electro-magnetic” duality group that acts on the 3-form field strength: in 4d this is $SO(2)$ that rotates (c_1, c_2) and in 6d this is Z_2 .

3.4. Symmetries and limits of the solution

Let us now discuss some properties of the solutions (3.8), (3.9) and (3.18), (3.19) corresponding to the metric (3.3).

A. Swap of the coordinates on the sphere

The metric (3.3) is invariant under swapping of the angles on S^3 together with a redefinition of r :

$$\psi \leftrightarrow \varphi, \quad r \rightarrow \sqrt{\frac{1-r^2}{1+\kappa^2 r^2}} \quad (3.21)$$

One can check that the scalar fields in (3.8), (3.9) remain invariant provided one also transforms a as

$$a \rightarrow \frac{1-a}{1+a\kappa^2}. \quad (3.22)$$

In particular, the points $a = 0$ and $a = 1$ are interchanged, and, in fact, the complete $a = 0$ solution (3.18) is interchanged with the $a = 1$ solution (3.19).

B. T-dualities

As in the $\text{AdS}_2 \times S^2$ case, we can perform a sequence of transformations:

1. T-dualize along t and φ directions.
2. Continue the new coordinates as $(t, \varphi) \rightarrow i(t, \varphi)$ and rescale $(\rho, r) \rightarrow \ell(\rho, r)$, $\ell \equiv \kappa^{-1}$. This sequence maps the $a = 0$ solution (3.18) back to itself (after an appropriate rescaling of coordinates). The $a = 1$ background (3.19) is mapped into a solution with imaginary fluxes, which cannot be made real by further analytic continuations.

C. Inversion of coordinates

As in the $\text{AdS}_2 \times S^2$ case, the limit $\kappa = \infty$ simplifies after a sequence of duality transformations and analytic continuations:

1. Rewrite (2.13)–(2.16) in terms of $x \equiv 1/\rho$ and $y \equiv 1/r$.
2. T-dualize along ψ and χ .
3. Define $\tilde{L} = -i\ell L$, $\ell \equiv \kappa^{-1}$.

One can show that then the RR fields become complex unless $a = 0$.

Thus as in the $\text{AdS}_2 \times S^2$ case the large- κ limit appears to prefer the $a = 0$ solution. At the same time, the $\kappa = i$ or pp-wave limit [4] appears to prefer the $a = 1$ solution. Namely, if we consider again the limit (2.27) then C_2 in (3.18) diverges, while C_2 in (3.19) remains finite and real, i.e.

$$a = 1: \quad C_2 = \frac{1}{(1+\rho^2)(1-r^2)} (\rho^2 d\chi + r^2 d\psi) \wedge dx^+. \quad (3.23)$$

Comparing this with Eqs. (3.28) and (3.29) in [4] giving the 2-form potential in this limit and accounting for the coordinate change we find a perfect match.

The value $a = 1$ for $\kappa = i$ is also singled out by comparing the corresponding limits of the dilatons in (3.18) and (3.19)

$$\begin{aligned} e^{-2\Phi} \Big|_{a=0} &= \frac{(1+\rho^2)(1-r^2)}{[1+(\rho r)^2]^2}, \\ e^{-2\Phi} \Big|_{a=1} &= \frac{(1+\rho^2)(1-r^2)}{(1+\rho^2)^2(1-r^2)^2} = \frac{1}{(1+\rho^2)(1-r^2)} \end{aligned} \quad (3.24)$$

with the expression for the natural value of the dilaton found directly in this limit in [4] (see Eqs. (3.16), (3.22) and (3.26) there).

We conclude that, as in the $\text{AdS}_2 \times \text{S}^2$ case, the limits $\kappa = \infty$ and $\kappa = i$ appear to select two different values of a , suggesting that there should exist an interpolating solution with $a = a(\kappa)$. While the six-dimensional Rainich conditions discussed in Section 3.2 rule out such a solution supported by a single 3-form flux, a preliminary investigation suggests that there may exist a 6d supergravity solution with two different 3-form fields being non-zero.

3.5. A generalization: two-parameter deformation

It was shown in [4] that the κ -deformation of the $\text{AdS}_3 \times \text{S}^3$ metric corresponds to a special case of the general two-parameter Fateev model [10] which is also the same as the two-parameter family of classically integrable bi-Yang–Baxter sigma models constructed in [11,13]. The corresponding deformed $\text{AdS}_3 \times \text{S}^3$ metric can be written as

$$\begin{aligned}
 ds^2 = & \frac{1}{F(\rho)} \left[-(1 + \rho^2)[1 + \kappa_-^2(1 + \rho^2)]dt^2 + \frac{d\rho^2}{1 + \rho^2} \right. \\
 & \left. + \rho^2(1 - \kappa_+^2\rho^2)d\chi^2 + 2\kappa_-\kappa_+\rho^2(1 + \rho^2)dt d\chi \right] \\
 & + \frac{1}{\tilde{F}(r)} \left[(1 - r^2)[1 + \kappa_-^2(1 - r^2)]d\varphi^2 + \frac{dr^2}{1 - r^2} \right. \\
 & \left. + r^2(1 + \kappa_+^2r^2)d\psi^2 + 2\kappa_+\kappa_-r^2(1 - r^2)d\psi d\varphi \right], \\
 F = & 1 + \kappa_-^2(1 + \rho^2) - \kappa_+^2\rho^2, \quad \tilde{F} = 1 + \kappa_-^2(1 - r^2) + \kappa_+^2r^2.
 \end{aligned} \tag{3.25}$$

For $\kappa_- = 0$, $\kappa_+ = \kappa$ we get back to the metric (3.3). There is no B -field. The supercoset model with this bosonic part was constructed in [14]. Similarly to the case of the κ -deformed $\text{AdS}_3 \times \text{S}^3$ metric, it should thus be possible to extend the metric (3.25) to a full supergravity solution.

Indeed, we found the following generalizations of the $a = 0$ (3.18) and $a = 1$ (3.19) solutions with both κ_+ and κ_- :

$$\begin{aligned}
 a = 0: \quad e^{-2\Phi} = & \frac{F(\rho)\tilde{F}(r)}{[P(\rho, r)]^2}, \quad P \equiv 1 + \kappa_-^2 - (\kappa_+^2 - \kappa_-^2)r^2\rho^2, \quad C = 0, \\
 C_2 = & \frac{\sqrt{1 + \kappa_-^2}}{P(\rho, r)} \left[(1 + \rho^2)dt \wedge d\chi + (1 - r^2)d\varphi \wedge d\psi \right. \\
 & + \kappa_+(1 + \rho^2)r^2dt \wedge d\psi - \kappa_+\rho^2(1 - r^2)d\chi \wedge d\varphi \\
 & \left. + \kappa_-(1 + \rho^2)(1 - r^2)dt \wedge d\varphi - \frac{\kappa_-(1 + \kappa_+^2)}{1 + \kappa_-^2}r^2\rho^2d\chi \wedge d\psi \right], \\
 a = 1: \quad e^{-2\Phi} = & \frac{F(\rho)\tilde{F}(r)}{[P(\rho, r)]^2}, \quad P \equiv 1 + \kappa_-^2 + (\kappa_+^2 - \kappa_-^2)(r^2 - \rho^2 + r^2\rho^2), \\
 C = & 0,
 \end{aligned} \tag{3.26}$$

$$\begin{aligned}
C_2 = \frac{\sqrt{1+\kappa_+^2}}{P(\rho, r)} & \left[\rho^2 dt \wedge d\chi - r^2 d\varphi \wedge d\psi + \kappa_- (1+\rho^2) r^2 dt \wedge d\psi \right. \\
& - \kappa_- \rho^2 (1-r^2) d\chi \wedge d\varphi + \frac{\kappa_+ (1+\kappa_-^2)}{1+\kappa_+^2} (1-r^2) (1+\rho^2) dt \wedge d\varphi \\
& \left. - \kappa_+ r^2 \rho^2 d\chi \wedge d\psi \right]. \quad (3.27)
\end{aligned}$$

As in the $\text{AdS}_2 \times \text{S}^2$ and $\text{AdS}_3 \times \text{S}^3$ cases discussed above, it is natural to expect that there should exist a one-parameter family of solutions including (3.26) and (3.27) as special cases.

Solutions (3.26) and (3.27) are interchanged by the transformation

$$\rho \rightarrow i\sqrt{1+\rho^2}, \quad r \rightarrow \sqrt{1-r^2}, \quad t \leftrightarrow \chi, \quad \varphi \leftrightarrow \psi, \quad \kappa_+ \leftrightarrow \kappa_-. \quad (3.28)$$

The invariance of the metric (3.25) under the map (3.28) was noted in [14].

4. Deformation of $\text{AdS}_5 \times \text{S}^5$

The extension of the κ -deformed $\text{AdS}_5 \times \text{S}^5$ metric and B -field [3] to a full supergravity solution turns out to be more challenging than in the above lower-dimensional cases. This is due, in particular, to the lack of isometries, i.e. a non-trivial dependence on the two extra angular coordinates. While we will not find a complete solution, in this section we shall discuss some of its features and draw analogies with the $\text{AdS}_2 \times \text{S}^2$ and $\text{AdS}_3 \times \text{S}^3$ cases.

Assuming a particular structure of the RR fluxes we shall find two different solutions to the scalar equations which are the counterparts of the $a=0$ (3.18) and $a=1$ (3.19) solutions in the $\text{AdS}_3 \times \text{S}^3$ case (we shall thus refer to them as the “ $a=0$ ” and “ $a=1$ ” solutions). To construct them it will be useful to switch to a T-dual frame where there is no B -field. We shall find that in this frame both solutions have vanishing RR scalar, $C=0$. However, in contrast to the $\text{AdS}_3 \times \text{S}^3$ case we have been unable to find a one-parameter family connecting these two special solutions. Moreover, the 10d algebraic Rainich conditions discussed in Section 3.2 imply that these solutions cannot be supported solely by a 5-form flux, i.e. one should excite other fluxes as well. We leave the study of this possibility for the future.

Our starting point will be the deformed $\text{AdS}_5 \times \text{S}^5$ metric and B -field corresponding [3] to the κ -deformed supercoset model of [2]⁸

$$\begin{aligned}
ds^2 = f(\rho) & \left[-(1+\rho^2) dt^2 + \frac{d\rho^2}{1+\rho^2} \right] + v(\rho, \zeta) \rho^2 (d\zeta^2 + c_\zeta^2 d\psi_1^2) + \rho^2 s_\zeta^2 d\psi_2^2 \\
& + \tilde{f}(r) \left[(1-r^2) d\varphi^2 + \frac{dr^2}{1-r^2} \right] + \tilde{v}(r, \theta) r^2 (d\theta^2 + c_\theta^2 d\phi_1^2) + r^2 s_\theta^2 d\phi_2^2, \quad (4.1) \\
B = \frac{1}{2} \kappa & [2v(\rho, \zeta) \rho^4 s_\zeta c_\zeta d\psi_1 \wedge d\zeta - 2\tilde{v}(r, \theta) r^4 s_\theta c_\theta d\phi_1 \wedge d\theta],
\end{aligned}$$

⁸ Let us mention that the detailed form of the model of [2] depends on a choice of the matrix R and there are several possibilities discussed in [7] (in the $\text{AdS}_3 \times \text{S}^3$ case there are two choices related to $\kappa_- = 0$ or $\kappa_+ = 0$ in (3.25), see [14]). Here we shall consider only the original choice in [2,3].

$$f = \frac{1}{1 - \kappa^2 \rho^2}, \quad \tilde{f} = \frac{1}{1 + \kappa^2 r^2}, \quad v = \frac{1}{1 + \kappa^2 \rho^4 s_\zeta^2}, \quad \tilde{v} = \frac{1}{1 + \kappa^2 r^4 s_\theta^2}, \quad (4.2)$$

where we used the shorthand notation $s_x = \sin x$, $c_x = \cos x$.

To put this background on the equal footing with the above low-dimensional solutions, it is convenient to remove the NS–NS B -field by performing the T-duality along the two compact coordinates ϕ_1 and ψ_1 which gives a non-diagonal metric

$$\begin{aligned} ds^2 = & f \left[-(1 + \rho^2) dt^2 + \frac{d\rho^2}{1 + \rho^2} \right] + v \rho^2 d\zeta^2 + \rho^2 s_\zeta^2 d\psi_2^2 + \tilde{f} \left[(1 - r^2) d\varphi^2 + \frac{dr^2}{1 - r^2} \right] \\ & + \tilde{v} r^2 d\theta^2 + r^2 s_\theta^2 d\phi_2^2 + \frac{1}{v \rho^2 c_\zeta^2} (d\psi_1 + k v \rho^4 s_\zeta c_\zeta d\zeta)^2 \\ & + \frac{1}{\tilde{v} r^2 c_\theta^2} (d\phi_1 - k \tilde{v} r^4 s_\theta c_\theta d\theta)^2. \end{aligned} \quad (4.3)$$

In the absence of the B -field the dilaton equation is

$$R + 4\nabla^2 \Phi - 4(\partial\Phi)^2 = 0. \quad (4.4)$$

We shall assume that in addition to the metric and the dilaton only the RR scalar C and the 5-form field $F_5 = C_4$ are excited. Then for a given metric the scalars C and Φ must satisfy an over-constrained system of the three equations – (4.4) as well as the RR scalar equation and the trace of the Einstein equation:

$$\nabla^2 C = 0, \quad R + 2\nabla^2 \Phi + 2e^{2\Phi} (\partial C)^2 = 0. \quad (4.5)$$

One of these three may be replaced with (cf. (A.7), (2.7), (3.4))

$$\nabla^2 (C^2 + e^{-2\Phi}) = 0. \quad (4.6)$$

Since the metric (4.3) was obtained from the deformed $\text{AdS}_5 \times S^5$ NS–NS background by the application of T-dualities, it should have a non-trivial dilaton even in the absence of the deformation (we shall denote the dilaton in T-dual frame with tilde)

$$e^{-2\tilde{\Phi}}|_{\kappa=0} = (\rho c_\zeta r c_\theta)^2. \quad (4.7)$$

In the general case we may then parametrize the dilaton as (cf. (2.14), (3.6))

$$e^{-2\tilde{\Phi}} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)(\rho c_\zeta r c_\theta)^2}{P_4(\rho, r, \zeta, \theta)}, \quad (4.8)$$

where P_4 is expected to have a polynomial dependence on ρ and r as well as a polynomial dependence on the trigonometric functions of ζ and θ .

Remarkably, as in the $\text{AdS}_3 \times S^3$ case (cf. (3.16), (3.17)), here we find two special solutions with $C = 0$:

$$a = 0: \quad e^{-2\tilde{\Phi}} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)(\rho r c_\theta c_\zeta)^2}{[1 - \kappa^2(\rho r)^2]^2 [1 - \kappa^2(\rho r s_\zeta s_\theta)^2]^2}, \quad C = 0, \quad (4.9)$$

$$\begin{aligned} a = 1: \quad e^{-2\tilde{\Phi}} = & \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)(\rho r c_\zeta c_\theta)^2}{[1 + \kappa^2 r^2 - \kappa^2(\rho s_\zeta)^2(1 - r^2)]^2 [1 - \kappa^2 \rho^2 + \kappa^2(r s_\theta)^2(1 + \rho^2)]^2}, \\ & C = 0. \end{aligned} \quad (4.10)$$

Undoing the T-duality, in the original frame (4.1), (4.2) the expressions for the dilaton become

$$a = 0: \quad e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)(1 + \kappa^2 \rho^2 \bar{\rho}^2)(1 + \kappa^2 r^2 \bar{r}^2)}{[1 - \kappa^2(\rho r)^2]^2 [1 - \kappa^2(\bar{\rho} \bar{r})^2]^2},$$

$$\bar{\rho} \equiv \rho s_\zeta, \quad \bar{r} \equiv r s_\theta, \quad (4.11)$$

$$a = 1: \quad e^{-2\Phi} = \frac{(1 - \kappa^2 \rho^2)(1 + \kappa^2 r^2)(1 + \kappa^2 \rho^2 \bar{\rho}^2)(1 + \kappa^2 r^2 \bar{r}^2)}{[1 + \kappa^2(r^2 - \bar{\rho}^2 + \bar{\rho}^2 r^2)]^2 [1 + \kappa^2(\bar{r}^2 - \rho^2 + \rho^2 \bar{r}^2)]^2}. \quad (4.12)$$

Let us now consider the $\kappa \rightarrow \infty$ and the $\kappa \rightarrow i$ limits [4] of the above expressions for the dilaton:

$\kappa = \infty$: As discussed in [4], in the $\kappa \rightarrow \infty$ limit the natural solution for the dilaton is expected to be a product of factors depending separately on the AdS_5 and S^5 coordinates. Taking $\kappa \rightarrow \infty$ in (4.11) we indeed find a factorization⁹

$$a = 0: \quad e^{-2\Phi} \Big|_{\kappa \rightarrow \infty} \rightarrow -\frac{1}{(\rho r s_\zeta s_\theta)^2}, \quad (4.13)$$

At the same time, the limit of the $a = 1$ expression (4.12) does not factorize for $\kappa \rightarrow \infty$.

$\kappa = i$: In the $\kappa \rightarrow i$ limit the dilaton may also be expected to factorize [4].¹⁰ However, this does not happen for the $a = 0$ expression (4.11). At the same time, the $\kappa \rightarrow i$ limit of the $a = 1$ dilaton (4.12) does factorize

$$a = 1: \quad e^{-2\Phi} \Big|_{\kappa \rightarrow i} \rightarrow \frac{(1 - \rho^4 s_\zeta^2)(1 - r^4 s_\theta^2)}{(1 + \rho^2)(1 - r^2)[1 + (\rho s_\zeta)^2]^2 [1 - (r s_\theta)^2]^2}. \quad (4.14)$$

Thus, as in the lower-dimensional cases, it seems natural to expect the existence of a one-parameter family of solutions with $a = a(\kappa)$ chosen so that $a(i) = 1$ and $a(\infty) = 0$.

At the same time, it is possible to check that the algebraic Rainich conditions (3.14) for existence of the F_5 flux are not satisfied by the stress tensor containing the contribution of the T-dual frame dilatons Eqs. (4.9) and (4.10) only. This indicates that one should look for more general solutions with several RR fields excited. This is analogous to our earlier observation that the a -family of scalar field solutions in the deformed $\text{AdS}_3 \times S^3$ case (3.8), (3.9) cannot be supported by just one 3-form RR field strength.

5. Some properties of the deformed backgrounds

While we did not find the full solution in the deformed $\text{AdS}_5 \times S^5$ case some of its properties are already evident from the form of the metric and the dilaton and are shared with the corresponding $\text{AdS}_2 \times S^2$ and $\text{AdS}_3 \times S^3$ solutions. Deformed backgrounds constructed in this paper represent a novel class of non-supersymmetric type IIB supergravity solutions which have factorized string-frame metric but non-factorized dilaton and RR fields.

For all the three deformed string-frame metrics (2.4), (3.3), (4.1) in dimensions 4, 6 and 10 there is a (naked) curvature singularity at $\rho = \kappa^{-1}$ (cf. (2.6)). The integrability of the underlying sigma models [2,9,10,12] implies, in particular, that it should be possible to find the explicit

⁹ The negative sign may be compensated by a formal imaginary constant shift of Φ .

¹⁰ The explicit form of the full “pp-wave” background corresponding to the $\kappa = i$ limit of the deformed $\text{AdS}_5 \times S^5$ solution was not found in [4].

form of the corresponding geodesics and study their approach to the singularity. At the same time, concentrating on the point-like limit may be misleading: one may need to investigate if the string probes “see” the singularity. For example, attempting to probe it with a long spinning folded string shows that the fold-points of the string remain at some finite distance from the singularity [20,21].¹¹

Since the string-frame dilaton equation is independent of the RR fluxes, the singularities of the dilaton are determined by the singularities of the metric and the NSNS B -field. From the dilaton equation (or the exact solutions (2.14), (3.8), (3.18), (3.19), (4.11), (4.12)) one concludes that near this point $e^\Phi \rightarrow (1 - \kappa^2 \rho^2)^{-1/2} \rightarrow \infty$. This means that the effective string coupling blows up, suggesting that one cannot study the near-singularity region using string perturbation theory.¹²

This conclusion may, however, be premature: due to lack of supersymmetry the leading-order supergravity solution may receive non-trivial α' corrections that may smear the singularity out in both the metric and the dilaton. Clarifying this issue requires a better understanding of the underlying deformed supercoset model at the quantum level.

It is interesting to note that while both the deformed metric g_{mn} and the dilaton Φ are singular, in all $\text{AdS}_n \times S^n$ cases the “T-duality invariant” volume density $e^{-2\Phi} \sqrt{-g}$ is regular at $\rho = \kappa^{-1}$. For example, if one performs a formal T-duality along the time t direction, in, e.g., (2.4) one gets a regular metric with a horizon at $\rho = \kappa^{-1}$ and with the T-dual string coupling e^Φ vanishing at that point. While this time-like T-duality is a formal transformation (the resulting type IIA background will have complex fluxes) this may be suggesting a hidden regularity of the original type IIB background.¹³

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Appendix A. Equations of motion and embedding into 10d supergravity

In Sections 2 and 3 we discussed $d = 4$ and $d = 6$ supergravities truncated to two scalar fields (the dilaton and the RR scalar) and one $d/2$ -form field. The corresponding actions may be written as

¹¹ For a discussion of classical string solutions in deformed geometry see also [22–24].

¹² From the 10d type IIB supergravity perspective, one may pass to the S-dual frame, where the dilaton is small near $\rho = \kappa^{-1}$. However, the metric will continue to be singular. Also, the S-dual solution will no longer have a deformed supercoset background interpretation.

¹³ One may draw an analogy with the T-duality in flat 2-space in polar coordinates or in Rindler space: a background $ds^2 = -r^{-2} dt^2 + dr^2$, $\Phi = -\ln r$ with curvature and dilaton singularity at $r = 0$ is T-dual to a regular one with $ds^2 = -r^2 dt^2 + dr^2$, $\Phi = 0$.

$$S = \int d^d x \sqrt{-g} \left[e^{-2\Phi} (R + 4(\partial\Phi)^2) - \frac{e_2}{4} F_{mn} F^{mn} - \frac{e_3}{12} F_{mnp} F^{mnp} - \frac{1}{2} (\partial C)^2 \right], \quad (\text{A.1})$$

where the coefficients (e_2, e_3) are

$$d = 4: \quad e_2 = 1, \quad e_3 = 0; \quad d = 6: \quad e_2 = 0, \quad e_3 = 1. \quad (\text{A.2})$$

The equations of motion coming from this action are

$$e^{-2\Phi} R_{mn} = -2e^{-2\Phi} \nabla_m \nabla_n \Phi + \frac{e_3}{4} \left(F_{mpq} F_n{}^{pq} - \frac{1}{6} g_{mn} F_{spq} F^{spq} \right) + \frac{e_2}{2} \left(F_{mk} F_n{}^k - \frac{1}{4} g_{mn} F^2 \right) + \frac{1}{2} \left[\partial_m C \partial_n C - \frac{1}{2} g_{mn} (\partial C)^2 \right], \quad (\text{A.3})$$

$$\nabla_m F^{mn} = 0, \quad \nabla_m F^{mnk} = 0, \quad (\text{A.4})$$

$$\left(-\nabla^2 + \frac{1}{4} R \right) e^{-\Phi} = 0, \quad \nabla^2 C = 0. \quad (\text{A.5})$$

It is convenient to separate the trace of the Einstein equation and combine it with the other scalar equations. Using the fact that the trace of the stress tensor of the $d/2$ -form in d dimensions vanishes, we have from (A.3)

$$e^{-2\Phi} R = -2e^{-2\Phi} \nabla^2 \Phi - \frac{d-2}{4} \partial_m C \partial^m C. \quad (\text{A.6})$$

Then (A.5) with (A.6) give

$$\nabla^2 \left(e^{-2\Phi} + \frac{d-2}{8} C^2 \right) = 0, \quad (\text{A.7})$$

which may be used in place of any of the three scalar equations in (A.5) and (A.6).

In the 10d case with non-vanishing B -field we get the following forms of the scalar equations

$$R - \frac{1}{4} H^2 + 2\nabla^2 \Phi + 2e^{2\Phi} (\partial C)^2 = 0, \quad \left[-\nabla^2 + \frac{1}{4} \left(R - \frac{1}{12} H^2 \right) \right] e^{-\Phi} = 0, \quad (\text{A.8})$$

$$\nabla^2 (e^{-2\Phi} + C^2) = \frac{1}{6} H^2, \quad \nabla^2 C = 0. \quad (\text{A.9})$$

Let us now review the embedding of the four- and six-dimensional systems (A.1), (A.2) in 10d supergravity.

The undeformed $\text{AdS}_3 \times S^3$ solution can be embedded in type IIB supergravity by identifying F_3 in (A.1) with RR 3-form field strength in ten dimensions. To embed the deformed 6d solution, we also identify C with the RR scalar in ten dimensions, i.e. the starting point is the following truncated 10d action

$$S = \int d^{10} x \sqrt{-g_{10}} \left[e^{-2\Phi} (R + 4(\partial\Phi)^2) - \frac{1}{12} F_{MNP} F^{MNP} - \frac{1}{2} (\partial C)^2 \right]. \quad (\text{A.10})$$

To perform the reduction to 6d, we write the ten-dimensional metric as¹⁴

$$ds_{10}^2 = g_{mn} dx^m dx^n + e^A dy_i dy_i, \quad (\text{A.11})$$

¹⁴ It is easy to check that more general warp factors on the torus, i.e. $\sum e^{A_i} dy_i dy_i$, do not lead to additional constraints for solutions with $A_i = A_j$, so that we may focus only on the volume mode.

where y_i are flat coordinates on T^4 . The standard dimensional reduction on the 4-torus then gives (see, e.g., [25])

$$S = \int d^6x \sqrt{-g} \left[e^{-2(\Phi-A)} \left(R + 4[\partial(\Phi-A)]^2 - (\partial A)^2 \right) - \frac{e^{2A}}{12} F_{mnp} F^{mnp} - \frac{e^{2A}}{2} (\partial C)^2 \right]. \quad (\text{A.12})$$

This reduces to (A.1) for $A = 0$, but equation of motion for A leads to an additional constraint:

$$\frac{1}{12} F_{mnp} F^{mnp} + \frac{1}{2} (\partial C)^2 = 0. \quad (\text{A.13})$$

This relation is satisfied by (3.18), (3.19), (3.26), (3.27).

The undeformed $\text{AdS}_2 \times S^2$ solution can be embedded in type II 10d supergravity in two different ways [15], which are related by T-dualities. In the absence of the Kalb–Ramond field, the action for type IIA supergravity is

$$S = \int d^{10}x \sqrt{-g_{10}} \left(e^{-2\Phi} [R + 4(\partial\Phi)^2] - \frac{1}{48} F_{MNPQ} F^{MNPQ} - \frac{1}{4} F_{MN} F^{MN} \right). \quad (\text{A.14})$$

Choosing the ansatz (z_i are 3 complex coordinates of 6-torus)

$$\begin{aligned} ds_{10}^2 &= g_{mn} dx^m dx^n + e^A dz_i d\bar{z}_i, & F^{(2)} &= \frac{1}{\sqrt{2}} \tilde{F}_{mn} dx^m \wedge dx^n, \\ F^{(4)} &= \frac{1}{\sqrt{2}} F_{mn} dx^m \wedge dx^n \wedge J_2 + \frac{1}{2} dC \wedge \text{Re } \Omega_3, \\ J_2 &\equiv \frac{i}{2} dz_k \wedge d\bar{z}_k, & \Omega_3 &\equiv dz_1 \wedge dz_2 \wedge dz_3, \end{aligned} \quad (\text{A.15})$$

and reducing on the 6-torus we find

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left[e^{-2\Phi+3A} \left(R + 4 \left[\partial \left(\Phi - \frac{3}{2} A \right) \right]^2 - \frac{3}{2} (\partial A)^2 \right) \right. \\ &\quad \left. - \frac{3e^A}{8} F_{mn} F^{mn} - \frac{e^{3A}}{8} \tilde{F}_{mn} \tilde{F}^{mn} - \frac{1}{2} (\partial C)^2 \right]. \end{aligned} \quad (\text{A.16})$$

To have a solution with $A = 0$, we must set

$$F_{mn} F^{mn} + \tilde{F}_{mn} \tilde{F}^{mn} = 0, \quad (\text{A.17})$$

and this constraint can be satisfied by imposing a relation

$$\tilde{F} = \star F. \quad (\text{A.18})$$

Substituting this relation for \tilde{F} into (A.16) and setting $A = 0$, we recover (A.1) with $e_2 = 1$, $e_3 = 0$.

The deformed $\text{AdS}_2 \times S^2$ solution can be also embedded into type IIB theory as

$$\begin{aligned} ds_{10}^2 &= g_{mn} dx^m dx^n + e^A dz_i d\bar{z}_i, \\ F^{(3)} &= \frac{1}{2} dC \wedge J_2 + \frac{1}{12} \star (dC \wedge J_2 \wedge J_2 \wedge J_2), \\ F^{(5)} &= \frac{1}{2} F_{mn} dx^m \wedge dx^n \wedge \text{Im } \Omega_3 - \frac{1}{2} [\star_4 (F_{mn} dx^m \wedge dx^n)] \wedge \text{Re } \Omega_3. \end{aligned} \quad (\text{A.19})$$

To write the action we relax the self-duality conditions by replacing F_5 with

$$F^{(5)} = \frac{1}{\sqrt{2}} F_{mn} dx^m \wedge dx^n \wedge \text{Im } \Omega_3. \quad (\text{A.20})$$

The dimensional reduction of the type IIB action

$$S = \int d^{10}x \sqrt{-g_{10}} \left[e^{-2\Phi} (R + 4(\partial\Phi)^2) - \frac{1}{12} F_{MNK} F^{MNK} - \frac{1}{480} F_{MNKLP} F^{MNKLP} \right], \quad (\text{A.21})$$

then gives

$$S = \int d^4x \sqrt{-g} \left[e^{-2\Phi+3A} \left(R + 4 \left[\partial \left(\Phi - \frac{3}{2} A \right) \right]^2 - \frac{3}{2} (\partial A)^2 \right) - \frac{1}{4} F_{mn} F^{mn} - \frac{3e^A}{8} (\partial C)^2 - \frac{e^{-3A}}{8} (\partial C)^2 \right]. \quad (\text{A.22})$$

This coincides with (A.1) for configurations with $A = 0$, and the equation of motion for A does not introduce additional constraints.

To summarize, we have demonstrated that the reduction of type II 10d supergravity reproduces the 4d action (A.1), but its 6d counterpart must be supplemented by the constraint (A.13).

Appendix B. Rainich conditions in four dimensions

As discussed in Section 3, to test whether a given stress–energy tensor can be sourced by a particular type of flux, we need a generalization of the Rainich condition to higher dimensions. To review the original condition in 4d, let us start with Maxwell stress tensor

$$T_m{}^n = F_{mk} F^{kn} - \frac{1}{4} \delta_m^n F_{sk} F^{ks}, \quad (\text{B.1})$$

which satisfies the two algebraic conditions

$$T_m{}^m = 0, \quad T_m{}^k T_k{}^n = \frac{1}{4} \delta_m^n T_s{}^k T_k{}^s. \quad (\text{B.2})$$

The first condition is obvious, while to prove the second one, we can go to the orthonormal frame and perform a (coordinate-dependent) rotation to put F_{mn} into a block-diagonal form

$$F_m{}^k = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & -a_2 & 0 \end{pmatrix}, \quad T_m{}^n = \frac{1}{2} (a_2^2 - a_1^2) \text{diag}(1, 1, -1, -1). \quad (\text{B.3})$$

It is useful to note that the Rainich conditions (B.2) imply that¹⁵

$$\text{tr } T = 0, \quad \text{tr } T^3 = 0. \quad (\text{B.4})$$

¹⁵ Here T^3 stands for $T_n{}^k T_k{}^m T_m{}^n$, etc.

Indeed, a 4×4 matrix T satisfies its own characteristic equation:

$$T^4 - \frac{1}{2}\text{tr}(T^2)T^2 - \frac{1}{3}\text{tr}(T^3)T + \det(T) = 0, \quad (\text{B.5})$$

where we used that $\text{tr} T = 0$. Then using (B.2) we conclude that $\text{tr}(T^3)T = 0$, implying (B.4).

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